

# A Note on Large Time Behavior of Velocity in the Barotropic Compressible Navier–Stokes Equations <sup>☆</sup>

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## Abstract

Recently, for periodic initial data with initial density allowed to vanish, Huang and Li [1] establish the global existence of strong and weak solutions for the two-dimensional compressible Navier–Stokes equations with no restrictions on the size of initial data provided the bulk viscosity coefficient is  $\lambda = \rho^\beta$  with  $\beta > 4/3$ . Moreover, the large-time behavior of the strong and weak solutions are also obtained, in which the velocity gradient strongly converges to zero in  $L^2$  norm. In this note, we further point out that the velocity strongly converges to an equilibrium velocity in  $H^1$  norm, in which the equilibrium velocity is uniquely determined by the initial data. Our result can also be regarded a correction for the result of large-time behavior of velocity in [2].

*Keywords:* Navier–Stokes equations, strong solution, weak solution, large time behavior.

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## 1. Introduction

In this note, we are concerned with the two-dimensional barotropic compressible Navier–Stokes equations which read as follows:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \quad (1.1)$$

$$\partial_t(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P(\rho) = \mu \Delta \mathbf{v} + \nabla((\lambda + \mu) \operatorname{div} \mathbf{v}), \quad (1.2)$$

where  $\rho$  and  $\mathbf{v}$  represent the density and velocity respectively, and the pressure  $P$  is given by

$$P(\rho) = a\rho^\gamma, \quad \gamma > 1.$$

Here  $a = e^S > 0$  is the constant determined by the entropy constant  $S$ , and  $\gamma \geq 1$  the adiabatic constant. Values of  $\gamma$  have their own physical significance, and are also take important part in the existence of solutions (see [3–6] for example). The viscosity coefficients satisfy the following hypothesis:

$$\mu = \text{constant}, \quad \lambda(\rho) = b\rho^\beta, \quad b > 0, \quad \beta > 0.$$

As in [1], we consider the Cauchy problem with the given initial density  $\rho_0$  and the given initial momentum  $\mathbf{m}_0$ , which are periodic with period 1 in each space direction  $x_i$ ,  $i = 1, 2$ , i.e., functions defined on  $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ . We require that

$$\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}), \quad \rho \mathbf{v}(\mathbf{x}, 0) = \mathbf{m}_0(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{T}^2.$$

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There is a huge literature concerning the theory of strong and weak solutions for the system of the multidimensional compressible Navier–Stokes with constant viscosity coefficients. The local existence and uniqueness of classical solutions are known in [7, 8] in the absence of vacuum and recently, for strong solutions also, in [9–11] for the case that the initial density need not be positive and may vanish in open sets. The global classical solutions were first obtained by Matsumura and Nishida [12] for initial data close to a non-vacuum equilibrium in some Sobolev space  $H^s$ . Later, Hoff [13] studied the problem for discontinuous initial data. For the existence of solutions for large data, the major breakthrough is due to Lions [14] (see also Feireisl [15, 16]), where he obtained global existence of weak solutions, defined as solutions with finite energy, when the exponent  $\gamma$  is suitably large. The main restriction on initial data is that the initial energy is finite, so that the density is allowed to vanish initially. Recently, Huang, Li and Xin [17] established the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible Navier–Stokes equations in the three-dimensional space with smooth initial data which are of small energy but possibly large oscillations; in particular, the initial density is allowed to vanish, even has compact support.

However, there are few results regarding global strong solvability for equations of multi-dimensional motions of viscous gas with no restrictions on the size of initial data. One of the first ever ones is due to Vaigant–Kazhikhov [18] who obtained a remarkable result which can be stated that the two-dimensional system (1.1)–(1.2) admits a unique global strong solution for large initial data away from vacuum provided  $\beta > 3$ . Lately, Perepelitsa [2] proved the global existence of a weak solution with uniform lower and upper bounds on the density, as well as the decay of the solution to an equilibrium state in a special case that

$$\beta > 3, \quad \gamma = \beta.$$

when the initial density is away from vacuum. Very recently, Jiu, Wang and Xin [19] consider classical solutions and removed the condition that the initial density should be away from vacuum in Vaigant–Kazhikhov [18] but still under the same condition that  $\beta > 3$  as that in [18]. No long after, Huang and Li establish the global existence of strong and weak solutions provided  $\beta > 4/3$  and  $\gamma > 1$ .

Before stating the exciting result of Huang and Li, we explain the notations and conventions used throughout this paper. We denote

$$\int f d\mathbf{x} = \int_{\mathbb{T}^2} f d\mathbf{x}, \quad \bar{f} = \frac{1}{|\mathbb{T}^2|} \int f d\mathbf{x}.$$

For  $1 \leq r \leq \infty$ , we also denote the standard Lebesgue and Sobolev spaces as follows:

$$L^r = L^r(\mathbb{T}^2), \quad W^{s,r} = W^{s,r}(\mathbb{T}^2), \quad H^s = W^{s,2}.$$

Then, we state the Huang and Li’s result concerning the global existence and large-time behavior of strong solutions as follows:

**Theorem 1.1.** *Assume that*

$$\beta > 4/3, \quad \gamma > 1, \tag{1.3}$$

*and that the initial data  $(\rho_0, \mathbf{m}_0)$  satisfy that for some  $q > 2$ ,*

$$0 \leq \rho_0 \in W^{1,q}, \quad \bar{\rho}_0 > 0, \quad \mathbf{v}_0 \in H^1, \quad \mathbf{m}_0 = \rho_0 \mathbf{v}_0.$$

Then the problem (1.1)–(1.2) has a unique global strong solution  $(\rho, \mathbf{v})$  satisfying

$$\begin{cases} \rho \in C([0, T], W^{1,1}), \quad \rho_t \in L^\infty(0, T; L^2), \\ \mathbf{v} \in L^\infty(0, T; H^1) \cap L^{(q+1)/q}(0, T; W^{2,q}), \\ t^{1/2}\mathbf{v} \in L^2(0, T; W^{2,q}), \quad t^{1/2}\mathbf{v}_t \in L^2(0, T; H^1), \\ \rho\mathbf{v} \in C([0, T], L^2), \quad \sqrt{\rho}\mathbf{v}_t \in L^2(\mathbb{T}^2 \times (0, T)), \end{cases}$$

for any  $0 < T < \infty$ . Moreover, if

$$\beta > 3/2, \quad 1 < \gamma < 3(\beta - 1), \quad (1.4)$$

there exists a constant  $C$  independent of  $T$  such that

$$\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^\infty} \leq C, \quad (1.5)$$

$$\sup_{0 \leq t \leq T} \|\mathbf{v}(\cdot, t)\|_{H^1} \leq C, \quad (1.6)$$

and the following large-time behavior holds:

$$\lim_{t \rightarrow \infty} (\|\rho - \bar{\rho}_0\|_{L^p} + \|\nabla \mathbf{v}\|_{L^2}) = 0, \quad (1.7)$$

for any  $p \in [1, \infty)$ .

**Remark 1.1.** The results above can be found in [1, Theorem 1.1], except for the estimate (1.6). Fortunately we can obtain (1.6) by [1, Proposition 3.5].

The result (1.7) above indicates that the density  $\rho(t)$  strongly converges to the equilibrium density  $\bar{\rho}_0$  in  $L^p$  norm as  $t \rightarrow \infty$ . We naturally propose an interesting question of whether there exists an equilibrium velocity  $\mathbf{v}_s$  such that the velocity  $\mathbf{v}(t)$  strongly converges  $\mathbf{v}_s$  in some norm as  $t \rightarrow \infty$ . In this note, we give the positive result. Next, we state our result, which will be proved in Section 2.

**Theorem 1.2.** Assume that the strong solution  $(\rho, \mathbf{v})$  is provided by Theorem 1.1. If  $(\rho, \mathbf{v})$  satisfies (1.6) and (1.7), then

$$\lim_{t \rightarrow \infty} \|\mathbf{v} - \mathbf{v}_s\|_{H^1} = 0, \quad (1.8)$$

where

$$\mathbf{v}_s := \frac{1}{\bar{\rho}_0 |\mathbb{T}^2|} \int \rho_0 \mathbf{v}_0 d\mathbf{x} \text{ is a constant vector.} \quad (1.9)$$

**Remark 1.2.** Assume that (1.3) holds and that the initial data  $(\rho_0, \mathbf{m}_0)$  satisfies that  $0 \leq \rho_0 \in L^\infty$ ,  $\mathbf{v}_0 \in H^1$ ,  $\mathbf{m}_0 = \rho_0 \mathbf{v}_0$ . Then the problem (1.1)–(1.2) possesses at least one global weak solution  $(\rho, \mathbf{v})$ . Moreover, if  $\beta$  and  $\gamma$  satisfy (1.4), there exists a constant  $C$  independent of  $T$  such that (1.5)–(1.7) hold true (see [1, Theorem 1.2]). We mention that such weak solution also satisfies (1.8).

**Remark 1.3.** When the initial density is away from vacuum, Perepelitsa [2] proved the global existence of a weak solution, as well as the convergence of the solution to an equilibrium state in a special case that  $\beta > 3$  and  $\gamma = \beta$ , where the author considered that equilibrium velocity is zero vector. According to Theorem 1.2, the equilibrium velocity is uniquely determined by the relation (1.9), and is not zero vector in the general case.

## 2. Proof of Theorem 1.2

In this section, we start to prove Theorem 1.2. First, exploiting (1.6) and the fact that  $H^1 \hookrightarrow L^2$  is compact, we have that for any sequence  $\{t_n\}_{n=1}^\infty \subset (0, \infty)$ , there exists a subsequence  $\{t_{n_m}\}_{m=1}^\infty \subset \{t_n\}_{n=1}^\infty$ , such that

$$\mathbf{v}(t_{n_m}) \rightharpoonup \mathbf{v}_M \text{ weakly in } H^1, \quad (2.1)$$

$$\mathbf{v}(t_{n_m}) \rightarrow \mathbf{v}_M \text{ strongly in } L^2, \quad (2.2)$$

$$n_m \rightarrow \infty \text{ as } m \rightarrow \infty.$$

Thanks to the condition (1.7), we see that  $\lim_{t \rightarrow \infty} \|\nabla \mathbf{v}(t)\| = 0$ , so  $\mathbf{v}_M$  must be a constant vector.

Next we shall show that the constant vector  $\mathbf{v}_M$  does not depend on the particular choice of subsequences. To this end, integrating the equation (1.2), we can deduce the momentum conservation

$$\int \rho(t) \mathbf{v}(t) d\mathbf{x} = \int \rho_0 \mathbf{v}_0 d\mathbf{x}. \quad (2.3)$$

Letting  $t := t_m \rightarrow \infty$  in (2.3), and using (1.7) and (2.2), then we obtain

$$\int \bar{\rho}_0 \mathbf{v}_M d\mathbf{x} = \lim_{t_m \rightarrow \infty} \int \rho(t_m) \mathbf{v}(t_m) d\mathbf{x} = \int \rho_0 \mathbf{v}_0 d\mathbf{x},$$

which yields

$$\mathbf{v}_M \equiv \mathbf{v}_s := \frac{\int \rho_0 \mathbf{v}_0 d\mathbf{x}}{\bar{\rho}_0 |\mathbb{T}^2|}. \quad (2.4)$$

Consequently, making use of the convergence of velocity in (1.7) and (2.2), we can conclude that for any sequence  $\{t_n\}_{n=1}^\infty \subset (0, \infty)$ , there exists a subsequence  $\{t_{n_m}\}_{m=1}^\infty \subset \{t_n\}_{n=1}^\infty$ , such that

$$\mathbf{v}(t_{n_m}) \rightarrow \mathbf{v}_s \text{ strongly in } H^1 \text{ as } m \rightarrow \infty. \quad (2.5)$$

Hence (1.8) holds, since the sequence  $\{t_n\}_{n=1}^\infty \subset (0, \infty)$  is arbitrary. This completes the proof of Theorem 1.2.

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